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FUNDAMENTAL SOLUTIONS IN UNSTEADY PROBLEMS OF ELECTROELASTICITY†

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Fundamental solutions of the unsteady linear problem of elasticity are constructed. The steady case was investigated earlier in [1].

1. Consider the unsteady equations of linear elasticity [2]

$$M_{ij}X_j + F_i = B_{ij}\ddot{X}_j, \quad B_{ij} = \rho\delta_{ij}(1-\delta_{4j}), \quad i,j = 1,.,4 \tag{1.1}$$

where $\{\chi_j\} = \{u_1, u_2, u_3, \phi\}$ is the generalized displacement vector and M_{ij} are second-order linear differential operators with constant coefficients

$$M_{ik} = C_{ijkl}\partial_j\partial_l, \quad i,k = 1,2,3, \quad M_{i4} = M_{4i} = e_{lij}\partial_j\partial_l, \quad i = 1,2,3 \tag{1.2}$$

$$M_{i4} = -\ni_{lj}\partial_j\partial_l$$

In relations (1.2) c_{ijkl} are the components of the elastic constants tensor, e_{lij} are the components of the piezoelectric constant tensor, $\ni_{lj}$ are the permittivities and ρ is the density. System (1.1) is non-hyperbolic (there is no term containing the second derivative with respect to time in the equation with $i = 4$), and its solutions possess a number of singularities.

We mean by a fundamental solution $\Psi_t^{(m)}(x, t)$ the solution of system (1.1) for $F_t = \delta_{im}\delta(x)\delta(t)$, where $\Psi^{(m)}_{i\,|\,t<0} = 0$.

To construct the fundamental solution $\Psi_i^{(m)}(x, t)$ we will apply a Fourier transformation to system (1.1) with respect to the coordinates and with respect to time, respectively, with the vector parameter $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ and the scalar parameter

$$\tilde{M}_{ij}(\alpha)\tilde{\Psi}_j^{(m)} - \omega^2 B_{ij}\tilde{\Psi}_j^{(m)} = -\delta_{im} \tag{1.3}$$

where $\tilde{M}_{ij}$ is obtained from M_{ij}, given by (1.2), by making the replacement $\partial_j \to (-i\alpha_j)$.

Solving system (1.3) for the transformant and carrying out an inverse Fourier transformation we obtain

$$\Psi_j^{(m)} = \frac{1}{(2\pi)^4}\int_{R_3}\int_{-\infty+i0}^{\infty+i0}\frac{p_{jm}(\alpha,\omega)}{p_0(\alpha,\omega)}e^{-i[(\alpha,x)+\omega t]}d\alpha d\omega \tag{1.4}$$

where $p_0(\alpha, \omega)$ is an eighth-order homogeneous polynomial and $p_{jm}(\alpha, \omega)$ are sixth-order homogeneous polynomials

$$p_{jm}(r\alpha, r\omega) = r^6 p_{jm}(\alpha,\omega), \quad p_0(r\alpha, r\omega) = r^8 p_0(\alpha,\omega) \tag{1.5}$$

The following assertions hold.

Lemma 1. We have the following asymptotic forms as $\omega \to \infty$

$$p_{44}(\alpha,\omega)\,/\,p_0(\alpha,\omega) = a_{44}(\alpha)(1 + O(\omega^{-2}))$$

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$p_{jm}(\alpha, \omega)/p_0(\alpha, \omega) = O(\omega^{-4})$ for the pairs $j, m = (1, 3), (2, 3)$ and those symmetrical to them, $p_{jm}(\alpha, \omega)/p_0(\alpha, \omega) = O(\omega^{-2})$ for the remaining j, m, where $a_{44}(\alpha)$ is a homogeneous function of order -2.

The proof of the lemma follows from the representation of the functions p_0 and p_{jm} in the form of fourth- and third-order determinants, respectively.

Lemma 2. The equation $p_0(\alpha, \omega) = 0$ has only real roots $\omega(\alpha)$.

In the anisotropic theory of elasticity a similar result follows from the presence of three positive roots in Christoffel's equation, which defines the propagation velocity of elastic waves in an anisotropic material. This result follows from the fact that the elastic energy is positive definite. In the case of an electroelastic material the internal energy is expressed as follows [2]:

$$U = \frac{1}{2} c_{ijkl} \varepsilon_{ij} \varepsilon_{kl} + \frac{1}{2} э_{ij} \phi_{,i} \phi_{,j}$$

and is a positive definite quadratic form. Here, by solving the last equation in (1.3) for $\tilde{\Psi}_4^{(4)}$, which is possible by virtue of the fact that the tensor $э_{ij}$ is positive definite and, substituting into the previous equations of (1.3), we obtain Christoffel's equation with "toughened" moduli. The assertion of the lemma therefore follows.

We will simplify representation (1.4) by replacing the variables $\Omega = \omega\gamma^{-1}$, $\gamma = |\alpha|$, $\alpha_k = \gamma\eta_k$, $|\eta| = 1$ and using Willis's method [3, 4]. The integral of γ will be understood in the sense of the theory of generalized functions [5]

$$I_n(z) = \lim_{\varepsilon \to 0} I_{n\varepsilon}(z), \quad I_{n\varepsilon}(z) = \int_0^\infty \gamma^{n-2} e^{-i\gamma z - \gamma\varepsilon} d\gamma, \quad n = 2, 3, \quad z = \Omega t + (\eta, x)$$
$$I_{2\varepsilon}(z) = \frac{1}{iz + \varepsilon}, \quad I_{3\varepsilon}(z) = i \frac{d}{dz} I_{2\varepsilon}(z) = -\frac{1}{(z - i\varepsilon)^2}, \quad \varepsilon > 0 \tag{1.6}$$

2. For simplicity we will consider the plane case ($n = 2$), $\eta_1 = \cos\phi$, $\eta_3 = \sin\phi$, $\phi \in [0, 2\pi]$. We have

$$\Psi_j^{(m)}(x, t) = -\frac{i}{(2\pi)^3} \lim_{\varepsilon \to 0} \int_{|\eta|=1} \int_{-\infty + i0}^{\infty + i0} \frac{p_{jm}(\eta, \Omega)}{p_0(\eta, \Omega)} \frac{d\eta d\Omega}{\Omega t + (\eta, x) - i\varepsilon} \tag{2.1}$$

We will analyse the integrand in (2.1). In the region $\operatorname{Im}\Omega > 0$ we have a single pole $\Omega = \Omega_\varepsilon t^{-1}[i\varepsilon - (\eta, x)]$ for $t > 0$ and no poles for $t < 0$. By virtue of Lemma 2 there are no other poles in the integrand in (2.1) when $\operatorname{Im}\Omega > 0$, and, by the theory of residues, we have the representation

$$\Psi_j^{(m)}(x, t) = \frac{t_+}{(2\pi)^2} \int_{|\eta|=1} \frac{p_{jm}(\eta t, (\eta, x))}{p_0(\eta t, (\eta, x))} d\eta, \quad t_+ = \begin{cases} t, & t > 0 \\ 0, & t \leq 0 \end{cases} \tag{2.2}$$

Here $j, m = 1, 2, 3, 4$, with the exception of the case $j = m = 4$.

By virtue of Lemma 1 the integrand in the expression for $\Psi_4^{(4)}$ is non-decreasing with respect to ω, and hence we cannot use the closure in the upper half-plane. Selecting the principal part, which is independent of ω

$$p_{44}(\alpha, \omega) / p_0(\alpha, \omega) = a_{44}(\alpha) + \tilde{p}_{44}(\alpha, \omega), \quad a_{44} = -(э_{lj}\alpha_l\alpha_j)^{-1} = -p(\alpha)$$

we obtain that

$$\Psi_4^{(4)}(x, t) = -\frac{1}{(2\pi)^2} \delta(t) \int_{R_2} p(\alpha) e^{-i(\alpha, x)} d\alpha + \frac{t_+}{(2\pi)^2} \int_{|\eta|=1} \tilde{p}_{44}(\eta t, (\eta, x)) d\eta \tag{2.3}$$

The first integral in (2.3) can be evaluated using the fundamental solution of the Laplace operator. We make the change of variables $\alpha_j = A_{jk} \upsilon_k$, where $A = \{A_{jk}\}$ is the inverse matrix to $э = \{э_{jk}\}$. Then

$$\frac{1}{(2\pi)^2} \int_{R_3} p(\alpha) e^{-i(\alpha, x)} d\alpha = \det A E(z), \quad E(z) = (\ln(z, z)) / 4\pi \tag{2.4}$$

and $E(x)$ is the known fundamental solution [5] ($\Delta E(x) = \delta(x)$), $z = э^{-1/2} x$. Thus, the fundamental solution

$\Psi_4'^{(4)}(x, t)$ has the form

$$\Psi_4'^{(4)}(x,t) = \frac{-\det \vartheta}{4\pi}\delta(t)\ln(z,z) + \frac{t_+}{(2\pi)^2}\int_{|\eta|=1} \tilde{p}_{44}(\eta t,(\eta,x))d\eta \tag{2.5}$$

Note the specific structure of this component of the fundamental solution, which contains the delta-function with respect to time, which corresponds to the instantaneous induction of potential over the whole region when there is electric excitation. This is a consequence of neglecting the derivative with respect to time in Maxwell's equation when describing the propagation of acoustic disturbances [4]. Note that, in other respects, the fundamental solutions (2.2) possess the usual properties of fundamental solutions for hyperbolic systems, in particular, for any instant of time t there is a region $x \in R_2$ where the perturbations are zero. This assertion is proved below

By virtue of Lemma $2 p_0(\eta, \Omega) = B_0(\eta)\Pi_{k=1}^{3}(\Omega - \lambda_k(\eta)), B_0(\eta) > 0, |\eta| = 1$, where $0 < \lambda_k^- \leqslant \lambda_k(\eta) \leqslant \lambda_k^+ (k = 1, 2, 3)$.

It is obvious that in the region

$$\max_{\substack{|\eta|=1\\ k=1,2,3}} (\eta, x)\lambda_k^{-1}(\eta) \geqslant t$$

the equation $p_0(\eta, (\eta, x)/t) = 0$ has no solutions. By virtue of this the integrand in (2.2) is regular and the integral (2.2) is zero. A similar assertion holds for the integral in (2.5).

Consider the structure of the integrands in the integrals over the unit circle in (2.2) and (2.5). To use these fundamental solutions in practice in the boundary-elements method [6] their integral representations are often sufficient, but we can also propose a method of evaluating them using the theory of residues. Taking into account the fact that the integrands in (2.2) and (2.5) are rational functions of $\sin 2\phi$ and $\cos 2\phi$, making the change of variable $v = \operatorname{ctg} \phi$ and putting $x_k t_+^{-1} = y_k$, we obtain

$$\int_{|\eta|=1} \frac{p_{jm}(\eta t,(\eta,y))}{p_0(\eta t,(\eta,y))}d\eta = 2\int_{-\infty}^{\infty} \frac{\Gamma_{jm}(v,y)}{\Gamma_0(v,y)}dv =$$

$$= 4\pi i\left(\sum_{\operatorname{Im} v_k > 0} \operatorname{res}\frac{\Gamma_{jm}}{\Gamma_0} + \frac{1}{2}\sum_{\operatorname{Im} v_k = 0} \operatorname{res}\frac{\Gamma_{jm}}{\Gamma_0}\right)$$

Here $\Gamma_0(v, y)$ is an eighth-degree polynomial, $\Gamma_{jm}(v, y)$ are sixth-degree polynomials, and v_k ($k = 1$–8) are the roots of Γ_0, which can only be calculated numerically for given y.

Note. For piezoelectric ceramics, polarized in the direction of the x_3 axis, which is the case most often encountered in practice, representations (2.2) and (2.5) remain true, but in this case $\Psi_2'^{(m)} \equiv 0$ and the polynomials p_{jm} and p_0 are of the fourth and sixth degree, respectively.

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